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GENERAL SOLUTION AND ULAM-HYERS STABILITY OF VIGINTI FUNCTIONAL EQUATIONS IN MULTI-BANACH SPACES

Ramdoss Murali*, Abasalt Bodaghi**, and Aruldass Antony Raj ***

Abstract. In this paper, we introduce the general form of a viginti functional equation. Then, we find the general solution and study the generalized Ulam-Hyers stability of such functional equation in multi-Banach spaces by using fixed point technique. Also, we indicate an example for non-stability case regarding to this new functional equation.

1. Introduction

The stability problem of functional equations has a long history. When we say the functional equation is stable, if for every approximate solution, there exists an exact solution near it. In 1940, Ulam [31] gave a wide-ranging talk before the mathematics club of the university of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms, which was first solved by Hyers [18]. In fact, he solved this stability problem for additive mappings subject to the Hyers' condition

$$\|f(x+y)-f(x)-f(y)\| \leq \delta$$

on approximately additive mappings $f : \mathcal{X} \longrightarrow \mathcal{Y}$ for a fixed $\delta \geq 0$ and all $x, y \in \mathcal{X}$ where $\mathcal{X}$ is a real normed space and $\mathcal{Y}$ a real Banach space. In 1950, Aoki [1] generalized the Hyers' theorem for additive mappings. In 1978, Th. M. Rassias [26] provided a generalized version of the Hyers' theorem which permitted the Cauchy difference to become

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Correspondence should be addressed to Abasalt Bodaghi, abasalt.bodaghi@gmail.com.

unbounded. By regarding a large influence of Ulam, Hyers and Rassias [26] on the investigation of stability problems of functional equations the stability phenomenon that was introduced and proved by Rassias is called the Ulam-Hyers-Rassias stability; some results regarding to the stability of various forms of the quadratic [12, 29], cubic [5, 9, 19, 23], quartic [3, 4, 5, 22], quintic [34], sextic [34], septic and octic [32], nonic [11, 24, 25], decic [2], undecic [27] and quattuordecic [28] functional equations have been investigated by a number of authors with more general domains and co-domains. For some stability of the mixed type functional equations refer to [6], [7], [8], [10], [17] and [21].

In this current work, we carry out the general solution and generalized Ulam-Hyers stability for the *viginti* functional equation

$$\begin{aligned} & f(x+10y) - 20f(x+9y) + 190f(x+8y) - 1140f(x+7y) \\ & + 4845f(x+6y) - 15504f(x+5y) + 38760f(x+4y) \\ & - 77520f(x+3y) + 125970f(x+2y) - 167960f(x+y) \\ & + 184756f(x) - 167960f(x-y) + 125970f(x-2y) \\ & - 77520f(x-3y) + 38760f(x-4y) - 15504f(x-5y) \\ & + 4845f(x-6y) - 1140f(x-7y) + 190f(x-8y) \\ & - 20f(x-9y) + f(x-10y) = 20!f(y) \end{aligned} \tag{1.1}$$

where $20! = 2432902008000000000$, in multi-Banach Spaces by applying fixed point technique. It is easily verified that that the function $f(x) = x^{20}$ satisfies the functional equation (1.1). In other words, every solution of the viginti functional equation is called a *viginti mapping*.

2. Notations and Preliminaries

We firstly recall some concepts regarding multi-Banach spaces setting. Let $(\mathcal{A}, \|\cdot\|)$ be a complex normed space, and let $k \in \mathbb{N}$. We denote by $\mathcal{A}^k$ the linear space $\mathcal{A} \oplus \mathcal{A} \oplus \mathcal{A} \oplus \cdots \oplus \mathcal{A}$ consisting of $k-$tuples $(x_1, \cdots, x_k)$ where $x_1, \cdots, x_k \in \mathcal{A}$. The linear operations on $\mathcal{A}^k$ are defined coordinate wise. The zero element of either $\mathcal{A}$ or $\mathcal{A}^k$ is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, 2, \cdots, k\}$ and by Ψ_k the group of permutations on k symbols.

Definition 2.1. [13] A multi-norm on $\left\{\mathcal{A}^k : k \in \mathbb{N}\right\}$ is a sequence $(\|\cdot\|) = (\|\cdot\|_k : k \in \mathbb{N})$ such that $\|\cdot\|_k$ is a norm on $\mathcal{A}^k$ for each $k \in \mathbb{N}$, $\|x\|_1 = \|x\|$ for each $x \in \mathcal{A}$, and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:

1. $\left\|\left(x_{\sigma(1)}, \cdots, x_{\sigma(k)}\right)\right\|_k = \|(x_1, \cdots, x_k)\|_k$ for $\sigma \in \Psi_k, x_1, \cdots, x_k \in \mathcal{A}$;
2. $\|(\alpha_1 x_1, \cdots, \alpha_k x_k)\|_k \le (\max_{i\in\mathbb{N}_k} |\alpha_i|) \|(x_1, \cdots, x_k)\|_k$ for $\alpha_1, \cdots, \alpha_k \in \mathbb{C}, \;\; x_1, \cdots, x_k \in \mathcal{A}$;
3. $\|(x_1, \cdots, x_{k-1}, 0)\|_k = \|(x_1, \cdots, x_{k-1})\|_{k-1}$ for $x_1, \cdots, x_{k-1} \in \mathcal{A}$;
4. $\|(x_1, \cdots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \cdots, x_{k-1})\|_{k-1}$ for $x_1, \cdots, x_{k-1} \in \mathcal{A}$.

In this case, we say that $\left((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N}\right)$ is a *multi-normed space.*

Let $k \in \mathbb{N}$. Suppose that $\left((\mathcal{A}^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi-normed spaces. We need the following two properties of multi-norms which are taken from [13].

(a) $\|(x, \cdots, x)\|_k = \|x\|$ for $x \in \mathcal{A}$,

(b) $\max_{i\in\mathbb{N}_k} \|x_i\| \le \|(x_1, \cdots, x_k)\|_k \le \sum_{i=1}^k \|x_i\| \le k \max_{i\in\mathbb{N}_k} \|x_i\|$, for all $x_1, \cdots, x_k \in \mathcal{A}$.

It follows from (b) that if $(\mathcal{A}, \|\cdot\|)$ is a Banach space, then $(\mathcal{A}^k, \|\cdot\|_k)$ is a Banach space for each $k \in \mathbb{N}$. In this case, $\left((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N}\right)$ is a multi-Banach space.

Lemma 2.2. [13] *Suppose that $k \in \mathbb{N}$ and $(x_1, \cdots, x_k) \in \mathcal{A}^k$. For each $j \in \{1, \cdots, k\}$, let $\left(x_n^j\right)_{n=1,2,\cdots}$ be a sequence in $\mathcal{A}$ such that $\lim_{n\to\infty} x_n^j = x_j$. Then*

$$\lim_{n\to\infty} \left(x_n^1 - y_1, \cdots, x_n^k - y_k\right) = (x_1 - y_1, \cdots, x_k - y_k) \tag{2.1}$$

holds for all $(y_1, \cdots, y_k) \in \mathcal{A}^k$.

Definition 2.3. [13] Let $\left((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N}\right)$ be a multi-normed space. A sequence (x_n) in $\mathcal{A}$ is a multi-null sequence if for each $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}} \|(x_n, \cdots, x_{n+k-1})\|_k \le \delta \qquad (n \ge n_0)\,. \tag{2.2}$$

Let $x \in \mathcal{A}$, we say that the sequence (x_n) is multi-convergent to x in $\mathcal{A}$ and write $\lim_{n\to\infty} x_n = x$ if $(x_n - x)$ is a multi-null sequence.

Here, we state the following theorem due to Margolis and Diaz which is useful to our purpose (an extension of the result was given in [30]).

Theorem 2.4. ([14] The fixed point alternative) *Let (Ω, d) be a complete generalized metric space and $\mathcal{J} : \Omega \longrightarrow \Omega$ be a mapping with Lipschitz constant $L < 1$. Then, for each element $x \in \Omega$, either*

$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$ *for all* $n \geq 0$, *or there exists a natural number* n_0 *such that*

(i) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty$ *for all* $n \geq n_0$;

(ii) *the sequence* $\{\mathcal{J}^n x\}$ *is convergent to a fixed point* y^* *of* $\mathcal{J}$;

(iii) y^* *is the unique fixed point of* $\mathcal{J}$ *in the set*

$$\Lambda = \{y \in \Omega : d(\mathcal{J}^{n_0} x, y) < \infty\};$$

(iv) $d(y, y^*) \leq \frac{1}{1-L} d(y, \mathcal{J} y)$ *for all* $y \in \Lambda$.

Throughout this paper, we use the abbreviation for the given mapping $f : \mathcal{X} \longrightarrow \mathcal{Y}$ as follows.

$$\begin{aligned}\mathcal{G}(x,y) =& f(x+10y) - 20f(x+9y) + 190f(x+8y) - 1140f(x+7y) \\ &+ 4845f(x+6y) - 15504f(x+5y) + 38760f(x+4y) \\ &- 77520f(x+3y) + 125970f(x+2y) - 167960f(x+y) \\ &+ 184756f(x) - 167960f(x-y) + 125970f(x-2y) \\ &- 77520f(x-3y) + 38760f(x-4y) - 15504f(x-5y) \\ &+ 4845f(x-6y) - 1140f(x-7y) + 190f(x-8y) \\ &- 20f(x-9y) + f(x-10y) - 20!f(y)\end{aligned}$$

for all $x, y \in \mathcal{X}$.

3. General Solution of (1.1)

In this section, we show that every solution of the equation (1.1) is a viginti map.

THEOREM 3.1. *Let* $f : \mathcal{A} \longrightarrow \mathcal{B}$ *be a mapping satisfying* (*1.1*) *for all* $x, y \in \mathcal{A}$. *Then,* f *is viginti.*

Proof. Putting $x = y = 0$ in (1.1), we have $f(0) = 0$. Substituting (x, y) by (x, x) in (1.1), we get

$$\begin{aligned}& f(11x) - 20f(10x) + 190f(9x) - 1140f(8x) + 4845f(7x) - 15504f(6x) \\ &\quad + 38760f(5x) - 77520f(4x) + 125970f(3x) - 167960f(2x) \\ &\quad + 184756f(x) + 125970f(-x) - 77520f(-2x) + 38760f(-3x) \\ &\quad - 15504f(-4x) + 4845f(-5x) - 1140f(-6x) + 190f(-7x) \\ &\quad - 20f(-8x) + f(-9x) - 20!f(x) = 0 \qquad (3.1)\end{aligned}$$

for all $x \in \mathcal{A}$. Interchanging (x, y) into $(x, -x)$ in (1.1), we obtain

$$\begin{aligned}
&f(-9x) - 20f(-8x) + 190f(-7x) - 1140f(-6x) + 4845f(-5x)\\
&- 15504f(-4x) + 38760f(-3x) - 77520f(-2x) + 125970f(-x)\\
&+ 184756f(-x) - 167960f(2x) + 125970f(3x) - 77520f(4x)\\
&+ 38760f(5x) - 15504f(6x) + 4845f(7x)\\
&- 1140f(8x) + 190f(9x) - 20f(10x) + f(11x) - 20!f(-x) = 0
\end{aligned} \tag{3.2}$$

for all $x \in \mathcal{A}$. Subtracting (3.2) from (3.1), we arrive at

$$f(-x) = f(x) \tag{3.3}$$

for all $x \in \mathcal{A}$. Hence, f is an even mapping. Considering (x, y) by $(0, 2x)$ in (1.1) and using (3.3), we achieve

$$\begin{aligned}
&f(20x) - 20f(18x) + 190f(16x) - 1140f(14x) + 4845f(12x)\\
&- 15504f(10x) + 38760f(8x) - 77520f(6x)\\
&+ 125970f(4x) - 1216451004 \times 10^9 f(2x) = 0
\end{aligned} \tag{3.4}$$

for all $x \in \mathcal{A}$. Replacing(x, y) by $(10x, x)$ in (1.1) and using (3.3), we obtain

$$\begin{aligned}
&f(20x) - 20f(19x) + 190f(18x) - 1140f(17x) + 4845f(16x)\\
&- 15504f(15x) + 38760f(14x) - 77520f(13x) + 125970f(12x)\\
&- 167960f(11x) + 184756f(10x) - 167960f(9x) + 125970f(8x)\\
&- 77520f(7x) + 38760f(6x) - 15504f(5x)\\
&+ 4845f(4x) - 1140f(3x) + 190f(2x) - 20!f(x) = 0
\end{aligned} \tag{3.5}$$

for all $x \in \mathcal{A}$. Subtracting (3.4) from (3.5), we find

$$\begin{aligned}
&20f(19x) - 210f(18x) + 1140f(17x) - 4655f(16x)\\
&+ 15504f(15x) - 39900f(14x) + 77520f(13x) - 121125f(12x)\\
&+ 167960f(11x) - 200260f(10x) + 167960f(9x) - 87210f(8x)\\
&+ 77520f(7x) - 116280f(6x) + 15504f(5x) + 121125f(4x)\\
&+ 1140f(3x) - 1216451004 \times 10^9 f(2x) + 20!f(x) = 0
\end{aligned} \tag{3.6}$$

for all $x \in \mathcal{A}$. Switching (x, y) into $(9x, x)$ in (1.1) and using (3.3), we reach

$$\begin{aligned}
&f(19x) - 20f(18x) + 190f(17x) - 1140f(16x) + 4845f(15x)\\
&- 15504f(14x) + 38760f(13x) - 77520f(12x) + 125970f(11x)
\end{aligned}$$

$$
\begin{aligned}
&- 167960f(10x) + 184756f(9x) - 167960f(8x) + 125970f(7x)\\
&- 77520f(6x) + 38760f(5x) - 15504f(4x) + 4845f(3x)\\
&- 1140f(2x) - 20!f(x) = 0
\end{aligned} \tag{3.7}
$$

for all $x \in \mathcal{A}$. Multiplying (3.7) by 20, we have

$$
\begin{aligned}
&20f(19x) - 400f(18x) + 3800f(17x) - 22800f(16x)\\
&+ 96900f(15x) - 310080f(14x) + 775200f(13x)\\
&- 1550400f(12x) + 2519400f(11x) - 3359200f(10x)\\
&+ 3695120f(9x) - 3359200f(8x) + 2519400f(7x)\\
&- 1550400f(6x) + 775200f(5x) - 310080f(4x)\\
&+ 96900f(3x) - 22800f(2x) - 20!(20)f(x) = 0
\end{aligned} \tag{3.8}
$$

for all $x \in \mathcal{A}$. Subtracting (3.6) and (3.8), we obtain the following result.

$$
\begin{aligned}
&190f(18x) - 2660f(17x) + 18145f(16x) - 81396f(15x)\\
&+ 270180f(14x) - 697680f(13x) + 1429275f(12x)\\
&- 2351440f(11x) + 3158940f(10x) - 3527160f(9x)\\
&+ 3271990f(8x) - 2441880f(7x) + 1434120f(6x)\\
&- 759696f(5x) + 431205f(4x)\\
&- 95760f(3x) - 1216451004 \times 10^9 f(2x) + 20!(21)f(x) = 0
\end{aligned} \tag{3.9}
$$

for all $x \in \mathcal{A}$. Replacing (x, y) by $(8x, x)$ in (1.1) and applying (3.3), we get

$$
\begin{aligned}
&f(18x) - 20f(17x) + 190f(16x) - 1140f(15x)\\
&+ 4845f(14x) - 15504f(13x) + 38760f(12x)\\
&- 77520f(11x) + 125970f(10x) - 167960f(9x)\\
&+ 184756f(8x) - 167960f(7x) + 125970f(6x)\\
&- 77520f(5x) + 38760f(4x) - 15504f(3x)\\
&+ 4846f(2x) - 20!f(x) = 0
\end{aligned} \tag{3.10}
$$

for all $x \in \mathcal{A}$. Multiplying (3.10) by 190, we arrive at

$$
\begin{aligned}
&190f(18x) - 3800f(17x) + 36100f(16x) - 216600f(15x)\\
&+ 920550f(14x) - 2945760f(13x) + 7364400f(12x)\\
&- 14728800f(11x) + 23934300f(10x) - 31912400f(9x)\\
&+ 35103640f(8x) - 31912400f(7x) + 23934300f(6x)
\end{aligned} \tag{3.11}
$$

$$\begin{aligned}&- 14728800f(5x) + 7364400f(4x) - 2945760f(3x)\\&+ 920740f(2x) - 20!(190)f(x) = 0\end{aligned}$$

for all $x \in \mathcal{A}$. Subtracting (3.9) from (3.11), we have

$$\begin{aligned}&1140f(17x) - 17955f(16x) + 135204f(15x) - 650370f(14x)\\&+ 2248080f(13x) - 5935125f(12x) + 12377360f(11x)\\&- 20775360f(10x) + 28385240f(9x) - 31831650f(8x)\\&+ 29470520f(7x) - 22500180f(6x) + 13969104f(5x)\\&- 6933195f(4x) + 2850000f(3x) - 1216451004 \times 10^9 f(2x)\\&+ 20!(211)f(x) = 0\end{aligned} \tag{3.12}$$

for all $x \in \mathcal{A}$. Interchanging (x, y) into $(7x, x)$ in (1.1) and also using (3.3), we obtain

$$\begin{aligned}&f(17x) - 20f(16x) + 190f(15x) - 1140f(14x)\\&+ 4845f(13x) - 15504f(12x) + 38760f(11x) - 77520f(10x)\\&+ 125970f(9x) - 167960f(8x) + 184756f(7x) - 167960f(6x)\\&+ 125970f(5x) - 77520f(4x) + 38761f(3x)\\&- 15524f(2x) - 20!f(x) = 0\end{aligned} \tag{3.13}$$

for all $x \in \mathcal{A}$. Multiplying (3.13) by 1140, we see that

$$\begin{aligned}&1140f(17x) - 22800f(16x) + 216600f(15x) - 1299600f(14x)\\&+ 5523300f(13x) - 17674560f(12x) + 44186400f(11x)\\&- 88372800f(10x) + 143605800f(9x) - 191474400f(8x)\\&+ 210621840f(7x) - 191474400f(6x) + 143605800f(5x)\\&- 88372800f(4x) + 44187540f(3x)\\&- 17697360f(2x) - 20!(1140)f(x) = 0\end{aligned} \tag{3.14}$$

for all $x \in \mathcal{A}$. Subtracting (3.12) from (3.14), we get

$$\begin{aligned}&4845f(16x) - 81396f(15x) + 649230f(14x) - 3275220f(13x)\\&+ 11739435f(12x) - 31809040f(11x) + 67597440f(10x)\\&- 115220560f(9x) + 159642750f(8x) - 181151320f(7x)\\&+ 168974220f(6x) - 129636696f(5x) + 81439605f(4x)\\&- 41337540f(3x) - 1216451004 \times 10^9 f(2x)\\&+ 20!(1351)f(x) = 0\end{aligned} \tag{3.15}$$

for all $x \in \mathcal{A}$. Replacing (x, y) into $(6x, x)$ in (1.1) and using (3.3), we achieve

$$\begin{aligned} & f(16x) - 20f(15x) + 190f(14x) - 1140f(13x) + 4845f(12x) \\ & - 15504f(11x) + 38760f(10x) - 77520f(9x) + 125970f(8x) \\ & - 167960f(7x) + 184756f(6x) - 167960f(5x) + 125971f(4x) \\ & - 77540f(3x) + 38950f(2x) - 20!f(x) = 0 \end{aligned} \tag{3.16}$$

for all $x \in \mathcal{A}$. Multiplying (3.16) by 4845, we derive

$$\begin{aligned} & 4845f(16x) - 96900f(15x) + 920550f(14x) - 5523300f(13x) \\ & + 23474025f(12x) - 75116880f(11x) + 187792200f(10x) \\ & - 375584400f(9x) + 610324650f(8x) - 813766200f(7x) \\ & + 895142820f(6x) - 813766200f(5x) + 610329495f(4x) \\ & - 375681300f(3x) + 188712750f(2x) - 20!(4845)f(x) = 0 \end{aligned} \tag{3.17}$$

for all $x \in \mathcal{A}$. Combining (3.15) and (3.17), we have

$$\begin{aligned} & 15504f(15x) - 271320f(14x) + 2248080f(13x) \\ & - 11734590f(12x) + 43307840f(11x) - 120194760f(10x) \\ & + 260363840f(9x) - 450681900f(8x) + 632614880f(7x) \\ & - 726168600f(6x) + 684129504f(5x) - 528889890f(4x) \\ & + 334343760f(3x) - 1216451004 \times 10^9 f(2x) \\ & + 20!(6196)f(x) = 0 \end{aligned} \tag{3.18}$$

for all $x \in \mathcal{A}$. Switching (x, y) into $(5x, x)$ in (1.1) and using (3.3), we obtain

$$\begin{aligned} & f(15x) - 20f(14x) + 190f(13x) - 1140f(12x) + 4845f(11x) \\ & - 15504f(10x) + 38760f(9x) - 77520f(8x) + 125970f(7x) \\ & - 167960f(6x) + 18475f(5x) - 167980f(4x) + 126160f(3x) \\ & - 78660f(2x) - 20!f(x) = 0 \end{aligned} \tag{3.19}$$

for all $x \in \mathcal{A}$. The relation (3.19) necessities that

$$\begin{aligned} & 15504f(15x) - 310080f(14x) + 2945760f(13x) \\ & - 17674560f(12x) + 75116880f(11x) - 240374016f(10x) \\ & + 600935040f(9x) - 1201870080f(8x) + 1953038880f(7x) \\ & - 2604051840f(6x) + 2864472528f(5x) - 2604361920f(4x) \\ & + 1955984640f(3x) - 1219544640f(2x) - 20!(15504)f(x) = 0 \end{aligned} \tag{3.20}$$

for all $x \in \mathcal{A}$. It follows from (3.18) and (3.20) that

$$\begin{aligned} &38760f(14x) - 697680f(13x) + 5939970f(12x) \\ &- 31809040f(11x) + 120179256f(10x) - 340571200f(9x) \\ &+ 751188180f(8x) - 1320424000f(7x) + 1877883240f(6x) \\ &- 2180343024f(5x) + 2075472030f(4x) - 1621640880f(3x) \\ &- 1216451003 \times 10^9 f(2x) + 20!(21700)f(x) = 0 \end{aligned} \tag{3.21}$$

for all $x \in \mathcal{A}$. Replacing (x, y) by $(4x, x)$ in (1.1) and applying (3.3), we have

$$\begin{aligned} &f(14x) - 20f(13x) + 190f(12x) - 1140f(11x) + 4845f(10x) \\ &- 15504f(9x) + 38760f(8x) - 77520f(7x) + 125971f(6x) \\ &- 167980f(5x) + 184946f(4x) - 169100f(3x) \\ &+ 130815f(2x) - 20!f(x) = 0 \end{aligned} \tag{3.22}$$

for all $x \in \mathcal{A}$. Multiplying (3.22) by 38760, we arrive at

$$\begin{aligned} &38760f(14x) - 775200f(13x) + 7364400f(12x) \\ &- 44186400f(11x) + 187792200f(10x) - 600935040f(9x) \\ &+ 1502337600f(8x) - 3004675200f(7x) + 4882635960f(6x) \\ &- 6510904800f(5x) + 7168506960f(4x) - 6554316000f(3x) \\ &+ 5070389400f(2x) - 20!(38760)f(x) = 0 \end{aligned} \tag{3.23}$$

for all $x \in \mathcal{A}$. The relations (3.21) and (3.23) imply that

$$\begin{aligned} &77520f(13x) - 1424430f(12x) + 12377360f(11x) \\ &- 67612944f(10x) + 260363840f(9x) - 751149420f(8x) \\ &+ 1684251200f(7x) - 3004752720f(6x) + 4330561776f(5x) \\ &- 5093034930f(4x) + 4932675120f(3x) \\ &- 1216451008 \times 10^9 f(2x) + 20!(60460)f(x) = 0 \end{aligned} \tag{3.24}$$

for all $x \in \mathcal{A}$. Interchanging (x, y) into $(3x, x)$ in (1.1) and using (3.3), we obtain

$$\begin{aligned} &f(13x) - 20f(12x) + 190f(11x) - 1140f(10x) + 4845f(9x) \\ &- 15504f(8x) + 38761f(7x) - 77540f(6x) \\ &+ 126160f(5x) - 169100(4x) + 189601f(3x) \\ &- 183464f(2x) - 20!f(x) = 0 \end{aligned} \tag{3.25}$$

for all $x \in \mathcal{A}$. Multiplying (3.25) by 77520, we reach

$$\begin{aligned}
&77520f(13x) - 1550400f(12x) + 14728800f(11x)\\
&- 88372800f(10x) + 375584400f(9x) - 1201870080f(8x)\\
&+ 3004752720f(7x) - 6010900800f(6x) + 9779923200f(5x)\\
&- 13108632000f(4x) + 14697869520f(3x) - 14222129280f(2x)\\
&- 20!(77520)f(x) = 0
\end{aligned} \tag{3.26}$$

for all $x \in \mathcal{A}$. Subtracting (3.24) from (3.26), we get

$$\begin{aligned}
&125970f(12x) - 2351440f(11x) + 20759856f(10x)\\
&- 115220560f(9x) + 450720660f(8x) - 1320501520f(7x)\\
&+ 3006148080f(6x) - 5449361424f(5x) + 8015597070f(4x)\\
&- 9765194400f(3x) - 1216450994 \times 10^9 f(2x)\\
&+ 20!(137980)f(x) = 0
\end{aligned} \tag{3.27}$$

for all $x \in \mathcal{A}$. Switching (x, y) into $(2x, x)$ in (1.1) and again applying the relation (3.3), we find

$$\begin{aligned}
&f(12x) - 20f(11x) + 190f(10x) - 1140f(9x) + 4846f(8x)\\
&- 15524f(7x) + 38950f(6x) - 78660f(5x) + 130815f(4x)\\
&- 183464f(3x) + 223516f(2x) - 20!f(x) = 0
\end{aligned} \tag{3.28}$$

for all $x \in \mathcal{A}$. Multiplying (3.28) by 125970, we have

$$\begin{aligned}
&125970f(12x) - 2519400f(11x) + 23934300f(10x)\\
&- 143605800f(9x) + 610450620f(8x) - 1955558280f(7x)\\
&+ 4906531500f(6x) - 9908800200f(5x) + 16478765550f(4x)\\
&- 23110960080f(3x) + 28156310520f(2x)\\
&- 20!(125970)f(x) = 0
\end{aligned} \tag{3.29}$$

for all $x \in \mathcal{A}$. We Subtract (3.27) from (3.29), and so

$$\begin{aligned}
&167960f(11x) - 3174444f(10x) + 28385240f(9x)\\
&- 159729960f(8x) + 635056760f(7x) - 1900383420f(6x)\\
&+ 4459438776f(5x) - 8463168480f(4x) + 13345765680f(3x)\\
&- 1216451022 \times 10^9 f(2x) + 20!(263950)f(x) = 0
\end{aligned} \tag{3.30}$$

for all $x \in \mathcal{A}$. It follows from (3.1) and (3.3) that

$$f(11x) - 20f(10x) + 191f(9x) - 1160f(8x) + 5035f(7x)$$

$$\begin{aligned}&-16644f(6x)+43605f(5x)-93024f(4x)+164730f(3x)\\&-245480f(2x)-20!f(x)=0\end{aligned}\tag{3.31}$$

for all $x \in \mathcal{A}$. Multiplying (3.31) by 167960, we arrive at

$$\begin{aligned}&167960f(11x)-3359200f(10x)+32080360f(9x)\\&-194833600f(8x)+845678600f(7x)-2795526240f(6x)\\&+7323895800f(5x)-15624311040f(4x)+276680508000f(3x)\\&-41230820800f(2x)-20!(167960)f(x)=0\end{aligned}\tag{3.32}$$

for all $x \in \mathcal{A}$. The relations (3.30) and (3.32) show that

$$\begin{aligned}&184756f(10x)-3695120f(9x)+35103640f(8x)\\&-210621840f(7x)+895142820f(6x)-2864457024f(5x)\\&+7161142560f(4x)-14322285120f(3x)\\&-1216450981\times 10^{9}f(2x)+20!(431910)f(x)=0\end{aligned}\tag{3.33}$$

for all $x \in \mathcal{A}$. Setting (x,y) by $(0,x)$ in (1.1) and also using (3.3), one can obtain

$$\begin{aligned}&f(10x)-20f(9x)+190f(8x)-1140f(7x)+4845f(6x)\\&-15504f(5x)+38760f(4x)-77520f(3x)+125970f(2x)\\&-\frac{20!}{2}(167960)f(x)=0\end{aligned}\tag{3.34}$$

for all $x \in \mathcal{A}$. Multiplying (3.34) by 184756, we get

$$\begin{aligned}&184756f(10x)-3695120f(9x)+35103640f(8x)\\&-210621840f(7x)+895142820f(6x)-2864457024f(5x)\\&+7161142560f(4x)-14322285120f(3x)\\&+23273713320f(2x)+20!(524288)f(x)=0\end{aligned}\tag{3.35}$$

for all $x \in \mathcal{A}$. Subtracting (3.33) from (3.35), we reach $f(2x) = 2^{20} f(x)$ for all $x \in \mathcal{A}$. Therefore, f is a viginti mapping. This completes the proof. □

4. Stability of (1.1) in Multi-Banach Spaces

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.1) in multi-Banach spaces by using the fixed point method (Theorem 2.4).

THEOREM 4.1. *Let $\mathcal{A}$ be an linear space and let $\big((\mathcal{B}^k, \|.\|_k) : k \in \mathbb{N}\big)$ be a multi-Banach space. Suppose that δ is a non-negative real number and $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a mapping fulfills*

$$\sup_{k\in\mathbb{N}} \|(\mathcal{G}f(x_1, y_1), \cdots, \mathcal{G}f(x_k, y_k))\|_k \leq \delta \tag{4.1}$$

$x_1, \cdots, x_k, y_1, \cdots, y_k \in \mathcal{A}$. Then, there exists a unique viginti mapping $\mathcal{V} : \mathcal{A} \longrightarrow \mathcal{B}$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - \mathcal{V}(x_1), \cdots, f(x_k) - \mathcal{V}(x_k))\|_k \leq \frac{61681}{1500635425 \times 10^{14}}\,\delta \tag{4.2}$$

for all $x_i \in \mathcal{A}$, where $i = 1, 2, \cdots, k$.

Proof. Taking $x_i = 0$ and changing y_i by $2x_i$ in (4.1), and dividing by 2 in the resulting equation, we arrive

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(f(20x_1) - 20f(18x_1) + 190f(16x_1) - 1140f(14x_1)\\
&+ 4845f(12x_1) - 15504f(10x_1) + 38760f(8x_1) - 77520f(6x_1)\\
&+ 125970f(4x_1) - 1216451004 \times 10^9 f(2x_1), \cdots, f(20x_k)\\
&- 20f(18x_k) + 190f(16x_k) - 1140f(14x_k) + 4845f(12x_k)\\
&- 15504f(10x_k) + 38760f(8x_k) - 77520f(6x_k) + 125970f(4x_k)\\
&- 1216451004 \times 10^9 f(2x_k)))\|_k \leq \frac{\delta}{2}
\end{aligned} \tag{4.3}$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Putting $x_1, \cdots, x_k$ into $10x_1, \cdots, 10x_k$ and switching $y_1, y_2, \cdots, y_k$ into $x_1, \cdots, x_k$, respectively in (4.1), we get

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \Big\|(f(20x_1) - 20f(19x_1) + 190f(18x_1) - 1140f(17x_1)\\
&+ 4845f(16x_1) - 15504f(15x_1) + 38760f(14x_1)\\
&- 77520f(13x_1) + 125970f(12x_1) - 167960f(11x_1)\\
&+ 184756f(10x_1) - 167960f(9x_1) + 125970f(8x_1)\\
&- 77520f(7x_1) + 38760f(6x_1) - 15504f(5x_1)\\
&+ 4845f(4x_1) - 1140f(3x_1) + 190f(2x_1) - 20!f(x_1), \cdots,\\
&f(20x_k) - 20f(19x_k) + 190f(18x_k) - 1140f(17x_k)\\
&+ 4845f(16x_k) - 15504f(15x_k) + 38760f(14x_k)
\end{aligned} \tag{4.4}$$

$$+4845f(4x_1)-1140f(3x_1)+190f(2x_1)-20!f(x_1),\cdots,$$
$$f(20x_k)-20f(19x_k)+190f(18x_k)-1140f(17x_k)$$
$$+4845f(16x_k)-15504f(15x_k)+38760f(14x_k)$$
$$-77520f(13x_k)+125970f(12x_k)-167960f(11x_k)$$
$$+184756f(10x_k)-167960f(9x_k)+125970f(8x_k)$$
$$-77520f(7x_k)+38760f(6x_k)-15504f(5x_k)$$
$$+4845f(4x_k)-1140f(3x_k)+190f(2x_k)-20!f(x_k))\Big\|_k\le\delta$$

for all $x_1,\cdots,x_k\in\mathcal{A}$. Combining (4.3) and (4.4), we have

$$\begin{aligned}
&\sup_{k\in\mathbb{N}}\|(20f(19x_1)-210f(18x_1)+1140f(17x_1)\\
&-4655f(16x_1)+15504f(15x_1)-39900f(14x_1)\\
&+77520f(13x_1)-121125f(12x_1)+167960f(11x_1)\\
&-200260f(10x_1)+167960f(9x_1)-87210f(8x_1)\\
&+77520f(7x_1)-116280f(6x_1)+15504f(5x_1)\\
&+121125f(4x_1)+1140f(3x_1)-1216451004\times10^9f(2x_1)\\
&+20!f(x_1),\cdots,20f(19x_k)-210f(18x_k)+1140f(17x_k)\\
&-4655f(16x_k)+15504f(15x_k)-39900f(14x_k)\\
&+77520f(13x_k)-121125f(12x_k)+167960f(11x_k)\\
&-200260f(10x_k)+167960f(9x_k)-87210f(8x_k)\\
&+77520f(7x_k)-116280f(6x_k)+15504f(5x_k)\\
&+121125f(4x_k)+1140f(3x_k)\\
&-1216451004\times10^9f(2x_k)+20!f(x_k))\|_k\le\frac{3}{2}\delta
\end{aligned}\tag{4.5}$$

for all $x_1,\cdots,x_k\in\mathcal{A}$. Replacing $x_1,\cdots,x_k$ by $9x_1,\cdots,9x_k$ and putting $y_1,y_2,\cdots,y_k$ by $x_1,\cdots,x_k$, respectively in (4.1) and using the evenness of f, we get

$$\sup_{k\in\mathbb{N}}\|(f(19x_1)-20f(18x_1)+190f(17x_1)$$
$$-1140f(16x_1)+4845f(15x_1)-15504f(14x_1)$$
$$+38760f(13x_1)-77520f(12x_1)+125970f(11x_1)$$
$$-167960f(10x_1)+184756f(9x_1)-167960f(8x_1)$$
$$+125970f(7x_1)-77520f(6x_1)+38760f(5x_1)$$

$$\begin{aligned}(4.6)\quad & - 15504f(4x_1) + 4845f(3x_1) - 1140f(2x_1)\\ & - 20!f(x_1), \cdots , f(19x_k) - 20f(18x_k) + 190f(17x_k)\\ & - 1140f(16x_k) + 4845f(15x_k) - 15504f(14x_k)\\ & + 38760f(13x_k) - 77520f(12x_k) + 125970f(11x_k)\\ & - 167960f(10x_k) + 184756f(9x_k) - 167960f(8x_k)\\ & + 125970f(7x_k) - 77520f(6x_k) + 38760f(5x_k)\\ & - 15504f(4x_k) + 4845f(3x_k) - 1140f(2x_k) - 20!f(x_k))\|_k \leq \frac{3}{2}\delta\end{aligned}$$

for all $x_1, \cdots , x_k \in \mathcal{A}$. Multiplying by 20 on both sides of (4.6), one can obtain

$$\begin{aligned}& \sup_{k\in\mathbb{N}} \|(20f(19x_1) - 400f(18x_1) + 3800f(17x_1)\\ & - 22800f(16x_1) + 96900f(15x_1) - 310080f(14x_1)\\ & + 775200f(13x_1) - 1550400f(12x_1) + 2519400f(11x_1)\\ & - 3359200f(10x_1) + 3695120f(9x_1) - 3359200f(8x_1)\\ & + 2519400f(7x_1) - 1550400f(6x_1) + 775200f(5x_1)\\ & - 310080f(4x_1) + 96900f(3x_1) - 22800f(2x_1)\\ (4.7)\quad & - 20!(20)f(x_1), \cdots , 20f(19x_k) - 400f(18x_k) + 3800f(17x_k)\\ & + 96900f(15x_k) - 310080f(14x_k) + 775200f(13x_k)\\ & - 1550400f(12x_k) - 22800f(16x_k) + 2519400f(11x_k)\\ & - 3359200f(10x_k) + 3695120f(9x_k) - 3359200f(8x_k)\\ & - 1550400f(6x_k) + 775200f(5x_k) - 310080f(4x_k)\\ & + 2519400f(7x_k) + 96900f(3x_k)\\ & - 22800f(2x_k) - 20!(20)f(x_k))\|_k \leq 20\delta\end{aligned}$$

for all $x_1, \cdots , x_k \in \mathcal{A}$. Subtracting (4.5) from (4.7), we find

$$\begin{aligned}& \sup_{k\in\mathbb{N}} \|(190f(18x_1) - 2660f(17x_1) + 18145f(16x_1)\\ & - 81396f(15x_1) + 270180f(14x_1) - 697680f(13x_1)\\ & + 1429275f(12x_1) - 2351440f(11x_1) + 3158940f(10x_1)\\ & - 3527160f(9x_1) + 3271990f(8x_1) - 2441880f(7x_1)\\ & + 1434120f(6x_1) - 759696f(5x_1) + 431205f(4x_1)\\ & - 95760f(3x_1) - 1216451004 \times 10^9 f(2x_1) + 20!(21)f(x_1)\end{aligned}$$

$$(4.8)\quad \begin{aligned}&, \cdots, 190f(18x_k) - 2660f(17x_k) + 18145f(16x_k)\\
&- 81396f(15x_k) + 270180f(14x_k) - 697680f(13x_k)\\
&+ 1429275f(12x_k) - 2351440f(11x_k) + 3158940f(10x_k)\\
&- 3527160f(9x_k) + 3271990f(8x_k) + 1434120f(6x_k)\\
&- 759696f(5x_k) + 431205f(4x_k) - 2441880f(7x_k)\\
&- 95760f(3x_k) - 1216451004 \times 10^9 f(2x_k)\\
&+ 20!(21)f(x_k))\|_k \leq \frac{43}{2}\delta\end{aligned}$$

$x_1, \cdots, x_k \in \mathcal{A}$. Setting $x_1, \cdots, x_k$ by $8x_1, \cdots, 8x_k$ and replacing $y_1, y_2, \cdots, y_k$ by $x_1, \cdots, x_k$, respectively in (4.1) and applying the evenness of f, we get

$$(4.9)\quad \begin{aligned}&\sup_{k\in\mathbb{N}} \|(f(18x_1) - 20f(17x_1) + 190f(16x_1) - 1140f(15x_1)\\
&+ 4845f(14x_1) - 15504f(13x_1) + 38760f(12x_1)\\
&- 77520f(11x_1) + 125970f(10x_1) - 167960f(9x_1)\\
&+ 184756f(8x_1) - 167960f(7x_1) + 125970f(6x_1)\\
&- 77520f(5x_1) + 38760f(4x_1) - 15504f(3x_1)\\
&+ 4846f(2x_1) - 20!f(x_1), \cdots, f(18x_k) - 20f(17x_k)\\
&+ 190f(16x_k) - 1140f(15x_k) + 4845f(14x_k)\\
&- 15504f(13x_k) + 38760f(12x_k) - 77520f(11x_k)\\
&+ 125970f(10x_k) - 167960f(9x_k) + 184756f(8x_k)\\
&- 167960f(7x_k) + 125970f(6x_k) - 77520f(5x_k)\\
&+ 38760f(4x_k) - 15504f(3x_k) + 4846f(2x_k) - 20!f(x_k))\|_k \leq \delta\end{aligned}$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Multiplying (4.9) by 190, we have

$$(4.10)\quad \begin{aligned}&\sup_{k\in\mathbb{N}} \|(190f(18x_1) - 3800f(17x_1) + 36100f(16x_1)\\
&- 216600f(15x_1) + 920550f(14x_1) - 2945760f(13x_1)\\
&+ 7364400f(12x_1) - 14728800f(11x_1) + 23934300f(10x_1)\\
&- 31912400f(9x_1) + 35103640f(8x_1) - 31912400f(7x_1)\\
&+ 23934300f(6x_1) - 14728800f(5x_1) + 7364400f(4x_1)\\
&- 2945760f(3x_1) + 920740f(2x_1) - 20!(190)f(x_1), \cdots,\\
&190f(18x_k) - 3800f(17x_k) + 36100f(16x_k) - 216600f(15x_k)\\
&+ 920550f(14x_k) - 2945760f(13x_k) + 7364400f(12x_k)\end{aligned}$$

$$
\begin{aligned}
&- 14728800f(11x_k) + 23934300f(10x_k) - 31912400f(9x_k) \\
&+ 35103640f(8x_k) - 31912400f(7x_k) + 23934300f(6x_k) \\
&- 14728800f(5x_k) + 7364400f(4x_k) - 2945760f(3x_k) \\
&+ 920740f(2x_k) - 20!(190)f(x_k))\|_k \leq 190\delta
\end{aligned}
$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Plugging (4.8) and (4.10), we see that

$$
\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(1140f(17x_1) - 17955f(16x_1) + 135204f(15x_1) \\
&- 650370f(14x_1) + 2248080f(13x_1) - 5935125f(12x_1) \\
&+ 12377360f(11x_1) - 20775360f(10x_1) + 28385240f(9x_1) \\
&- 31831650f(8x_1) + 29470520f(7x_1) - 22500180f(6x_1) \\
&+ 13969104f(5x_1) - 6933195f(4x_1) + 2850000f(3x_1) \\
&- 1216451004 \times 10^9 f(2x_1) + 20!(211)f(x_1), \cdots, \\
&1140f(17x_k) - 17955f(16x_k) + 135204f(15x_k) \\
&- 650370f(14x_k) + 2248080f(13x_k) - 5935125f(12x_k) \\
&+ 12377360f(11x_k) - 20775360f(10x_k) + 28385240f(9x_k) \\
&- 31831650f(8x_k) + 29470520f(7x_k) - 22500180f(6x_k) \\
&+ 13969104f(5x_k) - 6933195f(4x_k) + 2850000f(3x_k) \\
&- 1216451004 \times 10^9 f(2x_k) + 20!(211)f(x_k))\|_k \leq \frac{423}{2}\delta
\end{aligned} \tag{4.11}
$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Taking $x_1, \cdots, x_k$ by $7x_1, \cdots, 7x_k$ and interchanging $y_1, y_2, \cdots, y_k$ into $x_1, \cdots, x_k$, respectively in (4.1) and again using the evenness of f, we show that

$$
\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(f(17x_1) - 20f(16x_1) + 190f(15x_1) - 1140f(14x_1) \\
&+ 4845f(13x_1) - 15504f(12x_1) + 38760f(11x_1) \\
&- 77520f(10x_1) + 125970f(9x_1) - 167960f(8x_1) \\
&+ 184756f(7x_1) - 167960f(6x_1) + 125970f(5x_1) \\
&- 77520f(4x_1) + 38761f(3x_1) - 15524f(2x_1) \\
&- 20!f(x_1), \cdots, f(17x_k) - 20f(16x_k) + 190f(15x_k) \\
&- 1140f(14x_k) + 4845f(13x_k) - 15504f(12x_k) \\
&+ 38760f(11x_k) - 77520f(10x_k) + 125970f(9x_k) \\
&- 167960f(8x_k) + 184756f(7x_k) - 167960f(6x_k)
\end{aligned} \tag{4.12}
$$

$$+ 125970f(5x_k) - 77520f(4x_k) + 38761f(3x_k)$$
$$- 15524f(2x_k) - 20!f(x_k))\|_k \leq \delta$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Multiplying (4.12) by 1140, we achieve

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(1140f(17x_1) - 22800f(16x_1) + 216600f(15x_1) \\
&- 1299600f(14x_1) + 5523300f(13x_1) - 17674560f(12x_1) \\
&+ 44186400f(11x_1) - 88372800f(10x_1) + 143605800f(9x_1) \\
&- 191474400f(8x_1) + 210621840f(7x_1) - 191474400f(6x_1) \\
&+ 143605800f(5x_1) - 88372800f(4x_1) + 44187540f(3x_1) \\
&- 17697360f(2x_1) - 20!(1140)f(x_1), \cdots, \\
&1140f(17x_k) - 22800f(16x_k) + 216600f(15x_k) \\
&- 1299600f(14x_k) + 5523300f(13x_k) - 17674560f(12x_k) \\
&+ 44186400f(11x_k) - 88372800f(10x_k) + 143605800f(9x_k) \\
&- 191474400f(8x_k) + 210621840f(7x_k) - 191474400f(6x_k) \\
&+ 143605800f(5x_k) - 88372800f(4x_k) + 44187540f(3x_k) \\
&- 17697360f(2x_k) - 20!(1140)f(x_k))\|_k \leq 1140\delta
\end{aligned} \tag{4.13}$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Subtracting (4.11) from (4.13), we have

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(4845f(16x_1) - 81396f(15x_1) + 649230f(14x_1) \\
&- 3275220f(13x_1) + 11739435f(12x_1) \\
&- 31809040f(11x_1) + 67597440f(10x_1) \\
&- 115220560f(9x_1) + 159642750f(8x_1) \\
&- 181151320f(7x_1) + 168974220f(6x_1) \\
&- 129636696f(5x_1) + 81439605f(4x_1) \\
&- 41337540f(3x_1) - 1216451004 \times 10^9 f(2x_1) \\
&+ 20!(1351)f(x_1), \cdots, 4845f(16x_k) - 81396f(15x_k) \\
&+ 649230f(14x_k) - 3275220f(13x_k) + 11739435f(12x_k) \\
&- 31809040f(11x_k) + 67597440f(10x_k) - 115220560f(9x_k) \\
&+ 159642750f(8x_k) - 181151320f(7x_k) + 168974220f(6x_k) \\
&- 129636696f(5x_k) + 81439605f(4x_k) - 41337540f(3x_k) \\
&- 1216451004 \times 10^9 f(2x_k) + 20!(1351)f(x_k))\|_k \leq \frac{2703}{2}\delta
\end{aligned} \tag{4.14}$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Putting $x_1, \cdots, x_k$ into $6x_1, \cdots, 6x_k$ and replacing $y_1, y_2, \cdots, y_k$ by $x_1, \cdots, x_k$ in (4.1), respectively and using the evenness of f, we get

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(f(16x_1) - 20f(15x_1) + 190f(14x_1)\\
&- 1140f(13x_1) + 4845f(12x_1) - 15504f(11x_1)\\
&+ 38760f(10x_1) - 77520f(9x_1) + 125970f(8x_1)\\
&- 167960f(7x_1) + 184756f(6x_1) - 167960f(5x_1)\\
&+ 125971f(4x_1) - 77540f(3x_1) + 38950f(2x_1)\\
&- 20!f(x_1), \cdots, f(16x_k) - 20f(15x_k) + 190f(14x_k)\\
&- 1140f(13x_k) + 4845f(12x_k) - 15504f(11x_k)\\
&+ 38760f(10x_k) - 77520f(9x_k) + 125970f(8x_k)\\
&- 167960f(7x_k) + 184756f(6x_k) - 167960f(5x_k)\\
&+ 125971f(4x_k) - 77540f(3x_k)\\
&+ 38950f(2x_k) - 20!f(x_k))\|_k \leq \delta
\end{aligned} \tag{4.15}$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Multiplying (4.15) by 4845, we arrive at

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(4845f(16x_1) - 96900f(15x_1) + 920550f(14x_1)\\
&- 5523300f(13x_1) + 23474025f(12x_1) - 75116880f(11x_1)\\
&+ 187792200f(10x_1) - 375584400f(9x_1) + 610324650f(8x_1)\\
&- 813766200f(7x_1) + 895142820f(6x_1) - 813766200f(5x_1)\\
&+ 610329495f(4x_1) - 375681300f(3x_1) + 188712750f(2x_1)\\
&- 20!(4845)f(x_1), \cdots, 4845f(16x_k) - 96900f(15x_k)\\
&+ 920550f(14x_k) - 5523300f(13x_k) + 23474025f(12x_k)\\
&- 75116880f(11x_k) + 187792200f(10x_k) - 375584400f(9x_k)\\
&+ 895142820f(6x_k) - 813766200f(5x_k) + 610329495f(4x_k)\\
&+ 610324650f(8x_k) - 813766200f(7x_k) - 375681300f(3x_k)\\
&+ 188712750f(2x_k) - 20!(4845)f(x_k))\|_k \leq 4845\delta
\end{aligned} \tag{4.16}$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Subtracting (4.14) from (4.16), one can obtain

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(15504f(15x_1) - 271320f(14x_1)\\
&+ 2248080f(13x_1) - 11734590f(12x_1) + 43307840f(11x_1)\\
&- 120194760f(10x_1) + 260363840f(9x_1) - 450681900f(8x_1)
\end{aligned}$$

$$
\begin{aligned}
& + 632614880f(7x_1) - 726168600f(6x_1) + 684129504f(5x_1) \\
& - 528889890f(4x_1) + 334343760f(3x_1) \\
& - 1216451004 \times 10^9 f(2x_1) + 20!(6196)f(x_1), \cdots, \\
& 15504f(15x_k) - 271320f(14x_k) + 2248080f(13x_k) \\
& - 11734590f(12x_k) + 43307840f(11x_k) - 120194760f(10x_k) \\
& + 260363840f(9x_k) - 450681900f(8x_k) + 632614880f(7x_k) \\
& - 528889890f(4x_k) + 334343760f(3x_k) \\
& - 726168600f(6x_k) + 684129504f(5x_k) \\
& - 1216451004 \times 10^9 f(2x_k) + 20!(6196)f(x_k))\|_k \leq \frac{12393}{2}\delta
\end{aligned} \tag{4.17}
$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Switching $x_1, \cdots, x_k$ into $5x_1, \cdots, 5x_k$ and Changing $y_1, y_2, \cdots, y_k$ by $x_1, \cdots, x_k$, respectively in (4.1), we reach

$$
\begin{aligned}
& \sup_{k\in\mathbb{N}} \|(f(15x_1) - 20f(14x_1) + 190f(13x_1) - 1140f(12x_1) \\
& + 4845f(11x_1) - 15504f(10x_1) + 38760f(9x_1) \\
& - 77520f(8x_1) + 125970f(7x_1) - 167960f(6x_1) \\
& + 18475f(5x_1) - 167980f(4x_1) + 126160f(3x_1) \\
& - 78660f(2x_1) - 20!f(x_1), \cdots, f(15x_k) - 20f(14x_k) \\
& + 190f(13x_k) - 1140f(12x_k) + 4845f(11x_k) \\
& - 15504f(10x_k) + 38760f(9x_k) - 77520f(8x_k) + 125970f(7x_k) \\
& - 167960f(6x_k) + 18475f(5x_k) - 167980f(4x_k) \\
& + 126160f(3x_k) - 78660f(2x_k) - 20!f(x_k))\|_k \leq \delta
\end{aligned} \tag{4.18}
$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Multiplying (4.18) by 15504, we arrive

$$
\begin{aligned}
& \sup_{k\in\mathbb{N}} \|(15504f(15x_1) - 310080f(14x_1) + 2945760f(13x_1) \\
& - 17674560f(12x_1) + 75116880f(11x_1) - 240374016f(10x_1) \\
& + 600935040f(9x_1) - 1201870080f(8x_1) + 1953038880f(7x_1) \\
& - 2604051840f(6x_1) + 2864472528f(5x_1) - 2604361920f(4x_1) \\
& + 1955984640f(3x_1) - 1219544640f(2x_1) - 20!(15504)f(x_1) \\
& , \cdots, 15504f(15x_k) - 310080f(14x_k) + 2945760f(13x_k) \\
& - 17674560f(12x_k) + 75116880f(11x_k) - 240374016f(10x_k) \\
& + 600935040f(9x_k) - 1201870080f(8x_k) + 1953038880f(7x_k)
\end{aligned} \tag{4.19}
$$

$$- 2604051840f(6x_k) + 2864472528f(5x_k) - 2604361920f(4x_k)$$
$$+ 1955984640f(3x_k) - 1219544640f(2x_k)$$
$$- 20!(15504)f(x_k))\|_k \leq 155504\delta$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Subtracting (4.17) and (4.19)

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(15504f(15x_1) - 310080f(14x_1)\\
&+ 2945760f(13x_1) - 17674560f(12x_1) + 75116880f(11x_1)\\
&- 240374016f(10x_1) + 600935040f(9x_1) - 1201870080f(8x_1)\\
&+ 1953038880f(7x_1) - 2604051840f(6x_1) + 2864472528f(5x_1)\\
&- 2604361920f(4x_1) + 1955984640f(3x_1) - 1219544640f(2x_1)\\
&- 20!(15504)f(x_1), \cdots, 15504f(15x_k) - 310080f(14x_k)\\
&+ 2945760f(13x_k) - 17674560f(12x_k) + 75116880f(11x_k)\\
&- 240374016f(10x_k) + 600935040f(9x_k) - 1201870080f(8x_k)\\
&+ 1953038880f(7x_k) - 2604051840f(6x_k) + 2864472528f(5x_k)\\
&- 2604361920f(4x_k) + 1955984640f(3x_k) - 1219544640f(2x_k)\\
&- 20!(15504)f(x_k)\|_k \leq \frac{43401}{2}\delta
\end{aligned} \tag{4.20}$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Taking $x_1, \cdots, x_k$ by $4x_1, \cdots, 4x_k$ and changing $y_1, y_2, \cdots, y_k$ by $x_1, \cdots, x_k$, respectively in (4.1), we find

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(f(14x_1) - 20f(13x_1) + 190f(12x_1) - 1140f(11x_1)\\
&+ 4845f(10x_1) - 15504f(9x_1) + 38760f(8x_1) - 77520f(7x_1)\\
&+ 125971f(6x_1) - 167980f(5x_1) + 184946f(4x_1)\\
&- 169100f(3x_1) + 130815f(2x_1) - 20!f(x_1), \cdots, f(14x_k)\\
&- 20f(13x_k) + 190f(12x_k) - 1140f(11x_k) + 4845f(10x_k)\\
&- 15504f(9x_k) + 38760f(8x_k) - 77520f(7x_k) + 125971f(6x_k)\\
&- 167980f(5x_k) + 184946f(4x_k) - 169100f(3x_k)\\
&+ 130815f(2x_k) - 20!f(x_k))\|_k \leq \delta
\end{aligned} \tag{4.21}$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Multiplying (4.21) by 38760, we have

$$\sup_{k\in\mathbb{N}} \|(38760f(14x_1) - 775200f(13x_1) + 7364400f(12x_1)$$
$$- 44186400f(11x_1) + 187792200f(10x_1) - 600935040f(9x_1)$$
$$+ 1502337600f(8x_1) - 3004675200f(7x_1) + 4882635960f(6x_1)$$

$$
\begin{aligned}
&- 6510904800f(5x_1) + 7168506960f(4x_1) - 6554316000f(3x_1)\\
(4.22)\quad &+ 5070389400f(2x_1) - 20!(38760)f(x_1), \cdots,\\
&38760f(14x_k) - 775200f(13x_k) + 7364400f(12x_k)\\
&- 44186400f(11x_k) + 187792200f(10x_k) - 600935040f(9x_k)\\
&+ 1502337600f(8x_k) - 3004675200f(7x_k) + 4882635960f(6x_k)\\
&- 6510904800f(5x_k) + 7168506960f(4x_k) - 6554316000f(3x_k)\\
&+ 5070389400f(2x_k) - 20!(38760)f(x_k))\|_k \le 38760\delta
\end{aligned}
$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Subtracting (4.20) and (4.22)

$$
\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(77520f(13x_1) - 1424430f(12x_1)\\
&+ 12377360f(11x_1) - 67612944f(10x_1) + 260363840f(9x_1)\\
&- 751149420f(8x_1) + 1684251200f(7x_1) - 3004752720f(6x_1)\\
&+ 4330561776f(5x_1) - 5093034930f(4x_1) + 4932675120f(3x_1)\\
(4.23)\quad &- 1216451008 \times 10^9 f(2x_1) + 20!(60460)f(x_1), \cdots,\\
&77520f(13x_k) - 1424430f(12x_k) + 12377360f(11x_k)\\
&- 67612944f(10x_k) + 260363840f(9x_k) - 751149420f(8x_k)\\
&+ 1684251200f(7x_k) - 3004752720f(6x_k) + 4330561776f(5x_k)\\
&- 5093034930f(4x_k) + 4932675120f(3x_k)\\
&- 1216451008 \times 10^9 f(2x_k) + 20!(60460)f(x_k))\|_k \le \frac{120921}{2}\delta
\end{aligned}
$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Replacing $x_1, \cdots, x_k$ by $3x_1, \cdots, 3x_k$ and putting $y_1, y_2, \cdots, y_k$ by $x_1, \cdots, x_k$, respectively in (4.1) and also applying the evenness of f, we derive

$$
\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(f(13x_1) - 20f(12x_1) + 190f(11x_1)\\
&- 1140f(10x_1) + 4845f(9x_1) - 15504f(8x_1) + 38761f(7x_1)\\
&- 77540f(6x_1) + 126160f(5x_1) - 169100(4x_1) + 189601f(3x_1)\\
&- 183464f(2x_1) - 20!f(x_1), \cdots, f(13x_k) - 20f(12x_k)\\
&+ 190f(11x_k) - 1140f(10x_k) + 4845f(9x_k) - 15504f(8x_k)\\
&+ 38761f(7x_k) - 77540f(6x_k) + 126160f(5x_k) - 169100(4x_k)\\
(4.24)\quad &+ 189601f(3x_k) - 183464f(2x_k) - 20!f(x_k))\|_k \le \delta
\end{aligned}
$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Multiplying (4.24) by 77520, we get

$$
\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(77520f(13x_1) - 1550400f(12x_1) + 14728800f(11x_1)\\
&- 88372800f(10x_1) + 375584400f(9x_1) - 1201870080f(8x_1)\\
&+ 3004752720f(7x_1) - 6010900800f(6x_1) + 9779923200f(5x_1)\\
&- 13108632000f(4x_1) + 14697869520f(3x_1)\\
&- 14222129280f(2x_1) - 20!(77520)f(x_1), \cdots, 77520f(13x_k)\\
&- 1550400f(12x_k) + 14728800f(11x_k) - 88372800f(10x_k)\\
&+ 375584400f(9x_k) - 1201870080f(8x_k) + 3004752720f(7x_k)\\
&- 6010900800f(6x_k) + 9779923200f(5x_k)\\
&- 13108632000f(4x_k) + 14697869520f(3x_k)\\
&- 14222129280f(2x_k) - 20!(77520)f(x_k)\|_k \leq 77520\delta
\end{aligned} \tag{4.25}
$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Subtracting of (4.23) and (4.25) shows that

$$
\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(125970f(12x_1) - 2351440f(11x_1)\\
&+ 20759856f(10x_1) - 115220560f(9x_1)\\
&+ 450720660f(8x_1) - 1320501520f(7x_1)\\
&+ 3006148080f(6x_1) - 5449361424f(5x_1)\\
&+ 8015597070f(4x_1) - 9765194400f(3x_1)\\
&- 1216450994 \times 10^9 f(2x_1) + 20!(137980)f(x_1), \cdots,\\
&+ 20759856f(10x_k) - 115220560f(9x_k) + 450720660f(8x_k)\\
&125970f(12x_k) - 2351440f(11x_k) - 1320501520f(7x_k)\\
&+ 3006148080f(6x_k) - 5449361424f(5x_k)\\
&+ 8015597070f(4x_k) - 9765194400f(3x_k)\\
&- 1216450994 \times 10^9 f(2x_k) + 20!(137980)f(x_k))\|_k \leq \frac{275961}{2}\delta
\end{aligned} \tag{4.26}
$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Substituting $x_1, \cdots, x_k$ by $2x_1, \cdots, 2x_k$ and letting $y_1, y_2, \cdots, y_k$ by $x_1, \cdots, x_k$ in (4.1) and using the evenness of f, we find

$$
\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(f(12x_1) - 20f(11x_1) + 190f(10x_1) - 1140f(9x_1)\\
&+ 4846f(8x_1) - 15524f(7x_1) + 38950f(6x_1) - 78660f(5x_1)\\
&+ 130815f(4x_1) - 183464f(3x_1) + 223516f(2x_1) - 20!f(x_1)
\end{aligned}
$$

$$\begin{aligned}(4.27)\qquad &, \cdots, f(12x_k) - 20f(11x_k) + 190f(10x_k) - 1140f(9x_k)\\ &+ 4846f(8x_k) - 15524f(7x_k) + 38950f(6x_k)\\ &- 78660f(5x_k) + 130815f(4x_k) - 183464f(3x_k)\\ &+ 223516f(2x_k) - 20!f(x_k))\|_k \leq \delta\end{aligned}$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. If we multiply (4.27) by 125970, then we reach the following result.

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(125970f(12x_1) - 2519400f(11x_1)\\ &+ 23934300f(10x_1) - 143605800f(9x_1)\\ &+ 610450620f(8x_1) - 1955558280f(7x_1)\\ &+ 4906531500f(6x_1) - 9908800200f(5x_1)\\ &+ 16478765550f(4x_1) - 23110960080f(3x_1)\\ (4.28)\qquad &+ 28156310520f(2x_1) - 20!(125970)f(x_1)\\ &, \cdots, 125970f(12x_k) - 2519400f(11x_k) + 23934300f(10x_k)\\ &- 143605800f(9x_k) + 610450620f(8x_k)\\ &- 1955558280f(7x_k) + 4906531500f(6x_k)\\ &- 9908800200f(5x_k) + 16478765550f(4x_k)\\ &- 23110960080f(3x_k) + 28156310520f(2x_k)\\ &- 20!(125970)f(x_k))\|_k \leq 125970\delta\end{aligned}$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Plugging (4.26) into (4.28), we see that

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(167960f(11x_1) - 3174444f(10x_1) + 28385240f(9x_1)\\ &- 159729960f(8x_1) + 635056760f(7x_1)\\ &- 1900383420f(6x_1) + 4459438776f(5x_1)\\ &- 8463168480f(4x_1) + 13345765680f(3x_1)\\ (4.29)\qquad &- 1216451022 \times 10^9 f(2x_1) + 20!(263950)f(x_1), \cdots,\\ &167960f(11x_k) - 3174444f(10x_k) + 28385240f(9x_k)\\ &- 159729960f(8x_k) + 635056760f(7x_k) - 1900383420f(6x_k)\\ &+ 4459438776f(5x_k) - 8463168480f(4x_k)\\ &+ 13345765680f(3x_k) - 1216451022 \times 10^9 f(2x_k)\\ &+ 20!(263950)f(x_k))\|_k \leq \frac{527901}{2}\delta.\end{aligned}$$

Taking $x_1, \cdots, x_k$ by $x_1, \cdots, x_k$ and changing $y_1, y_2, \cdots, y_k$ by $x_1, \cdots, x_k$, respectively in (4.1) and again applying the evenness of f, we get

$$
\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(f(11x_1) - 20f(10x_1) + 191f(9x_1) - 1160f(8x_1) \\
&+ 5035f(7x_1) - 16644f(6x_1) + 43605f(5x_1) - 93024f(4x_1) \\
&+ 164730f(3x_1) - 245480f(2x_1) - 20!f(x_1), \cdots, f(11x_k) \\
&- 20f(10x_k) + 191f(9x_k) - 1160f(8x_k) + 5035f(7x_k) \\
&- 16644f(6x_k) + 43605f(5x_k) - 93024f(4x_k) \\
&+ 164730f(3x_k) - 245480f(2x_k) - 20!f(x_k))\|_k \leq \delta
\end{aligned} \tag{4.30}
$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Multiplying (4.30) by 167960, we arrive

$$
\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(167960f(11x_1) - 3359200f(10x_1) \\
&+ 32080360f(9x_1) - 194833600f(8x_1) + 845678600f(7x_1) \\
&- 2795526240f(6x_1) + 7323895800f(5x_1) \\
&- 15624311040f(4x_1) + 276680508000f(3x_1) \\
&- 20!(167960)f(x_1), \cdots, 167960f(11x_k) \\
&- 3359200f(10x_k) + 32080360f(9x_k) \\
&- 194833600f(8x_k) + 845678600f(7x_k) \\
&- 2795526240f(6x_k) + 7323895800f(5x_k) \\
&- 15624311040f(4x_k) + 276680508000f(3x_k) \\
&- 41230820800f(2x_k) - 20!(167960)f(x_k))\|_k \leq 167960\delta
\end{aligned} \tag{4.31}
$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Subtracting (4.29) and (4.31), we have

$$
\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(184756f(10x_1) - 3695120f(9x_1) \\
&+ 35103640f(8x_1) - 210621840f(7x_1) \\
&+ 895142820f(6x_1) - 2864457024f(5x_1) \\
&+ 7161142560f(4x_1) - 14322285120f(3x_1) \\
&- 1216450981 \times 10^9 f(2x_1) + 20!(431910)f(x_1), \cdots, \\
&184756f(10x_k) - 3695120f(9x_k) + 35103640f(8x_k) \\
&- 210621840f(7x_k) + 895142820f(6x_k) - 2864457024f(5x_k) \\
&+ 7161142560f(4x_k) - 14322285120f(3x_k) \\
&- 1216450981 \times 10^9 f(2x_k) + 20!(431910)f(x_k))\|_k \leq \frac{863821}{2}\delta
\end{aligned} \tag{4.32}
$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Taking $x_1 = x_2 =, \cdots, = x_k = 0$ and Changing $y_1, y_2, \cdots, y_k$ by $x_1, \cdots, x_k$ (4.1) and using (3.3), we obtain

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(f(10x_1) - 20f(9x_1) + 190f(8x_1) - 1140f(7x_1)\\
&+ 4845f(6x_1) - 15504f(5x_1) + 38760f(4x_1) - 77520f(3x_1)\\
&+ 125970f(2x_1) - \frac{20!}{2}(167960)f(x_1), \cdots,\\
&f(10x_k) - 20f(9x_k) + 190f(8x_k) - 1140f(7x_k) + 4845f(6x_k)\\
&- 15504f(5x_k) + 38760f(4x_k) - 77520f(3x_k) + 125970f(2x_k)\\
&- \frac{20!}{2}(167960)f(x_k))\|_k \leq \frac{\delta}{2}
\end{aligned} \tag{4.33}$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Multiplying (4.33) by 184756, we arrive at

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(184756f(10x_1) - 3695120f(9x_1)\\
&+ 35103640f(8x_1) - 210621840f(7x_1)\\
&+ 895142820f(6x_1) - 2864457024f(5x_1)\\
&+ 7161142560f(4x_1) - 14322285120f(3x_1)\\
&+ 23273713320f(2x_1) + 20!(524288)f(x_1), \cdots, 184756f(10x_k)\\
&- 3695120f(9x_k) + 35103640f(8x_k) - 210621840f(7x_k)\\
&+ 895142820f(6x_k) - 2864457024f(5x_k) + 7161142560f(4x_k)\\
&- 14322285120f(3x_k) + 23273713320f(2x_k)\\
&+ 20!(524288)f(x_k))\|_k \leq \frac{184756}{2}\delta
\end{aligned} \tag{4.34}$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Plugging (4.32) into (4.34), we obtain

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(-1216451004 \times 10^9 f(2x_1)\\
&+ 1275541328 \times 10^{15} f(x_1), \cdots, -1216451004 \times 10^9 f(2x_k)\\
&+ 1275541328 \times 10^{15} f(x_k))\|_k \leq \frac{1048577}{2}\delta
\end{aligned} \tag{4.35}$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Dividing on both sides by -1216451×10^9 in (4.35), we derive

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(-f(2x_1) + 1048576f(x_1), \cdots,\\
&- f(2x_k) + 1048576f(x_k))\|_k \leq \frac{1048577}{20!}\delta
\end{aligned} \tag{4.36}$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Once more, dividing on both sides by 1048576 in (4.36), we get

$$\sup_{k\in\mathbb{N}} \Big\|\big(\frac{1}{1048576} f(2x_1) - f(x_1), \cdots, \frac{1}{1048576} f(2x_k) - f(x_k)\big)\Big\|_k \leq \frac{1048577}{20!(1048576)}\delta \tag{4.37}$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Let $\Lambda = \{l : \mathcal{A} \longrightarrow \mathcal{B} | l(0) = 0\}$ and consider the generalized metric d defined on λ by

$$d(l,m) = \inf \Big\{\lambda \in [0,\infty] \Big| \sup_{k\in\mathbb{N}} \Big\| l(x_1) - m(x_1), \cdots, l(x_k) - m(x_k) \Big\|_k \leq \lambda \forall x_1, \cdots, x_k \in \mathcal{A}\Big\}.$$

It is easy to show that (Λ, d) is a generalized complete metric space (see [20]). We define an operator $\mathcal{J} : \Lambda \longrightarrow \Lambda$ by $\mathcal{J}l(x) = \frac{1}{2^{20}} l(2x)$ $(\forall x \in \mathcal{A})$. We assert that $\mathcal{J}$ is a strictly contractive operator. Given $l, m \in \Lambda$, let $\lambda \in [0, \infty]$ be an arbitary constant with $d(l,m) \leq \lambda$. From the definition if follows that

$$\sup_{k\in\mathbb{N}} \|l(x_1) - m(x_1), \cdots, l(x_k) - m(x_k)\|_k \leq \lambda \quad x_1, \cdots, x_k \in \mathcal{A}.$$

Therefore

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(\mathcal{J}l(x_1) - \mathcal{J}m(x_1), \cdots, \mathcal{J}l(x_k) - \mathcal{J}m(x_k))\|_k \\
&\quad\leq \sup_{k\in\mathbb{N}} \left\| \left(\frac{1}{2^{20}} l(2x_1) - \frac{1}{2^{20}} m(2x_1), \cdots, \frac{1}{2^{20}} l(2x_k) - \frac{1}{2^{20}} m(2x_k) \right) \right\|_k \\
&\quad\leq \frac{1}{2^{20}}\lambda
\end{aligned}$$

for $x_1, \cdots, x_k \in \mathcal{A}$. So, $d(\mathcal{J}l, \mathcal{J}m) \leq \frac{1}{2^{20}}\lambda d(\mathcal{J}l, \mathcal{J}m) \leq \frac{1}{2^{20}} d(l,m)$ for all $l, m \in \Lambda$. This means that $\mathcal{J}$ is strictly contractive operator on Λ with the Lipschitz constant $L = \frac{1}{2^{20}}$. By (4.37), we have $d(\mathcal{J}f, f) \leq \frac{1048577}{20!(1048576)}\delta$. Applying Theorem 2.4, we deduce the existence of a fixed point of $\mathcal{J}$ which is the existence of mapping $\mathcal{V} : \mathcal{A} \longrightarrow \mathcal{B}$ such that $\mathcal{V}(2x) = 2^{20}\mathcal{V}(x)$ for all $x \in \mathcal{A}$. Moreover, we have $d(\mathcal{J}^n f, \mathcal{V}) \to 0$, as $n \to \infty$ which necessities

$$\mathcal{V}(x) = \lim_{n\to\infty} \mathcal{J}^n f(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{20n}}$$

for all $x \in \mathcal{A}$. Also, $d(f, \mathcal{V}) \leq \frac{1}{1-\mathcal{L}} d(\mathcal{J}f, f)$ implies the inequality $d(f, \mathcal{V}) \leq \frac{1}{1-\frac{1}{2^{20}}} d(\mathcal{J}f, f) \leq \frac{61681}{1500635425 \times 10^{14}} \delta$. Put $x_1 = \cdots = x_k = 2^n x, y_1 = \cdots = y_k = 2^n y$ in (1.1) and divide both sides by 2^{20n}. Now, by applying the property (a) of multi-norms, we obtain

$$\|\mathcal{G}\mathcal{V}(x,y)\| = \lim_{n\to\infty} \frac{1}{2^{20n}} \|\mathcal{G}f(2^n x, 2^n y)\| \leq \lim_{n\to\infty} \frac{1}{2^{20n}} = 0$$

for all $x, y \in \mathcal{A}$. The uniqueness of $\mathcal{V}$ follows from the fact that $\mathcal{V}$ is the uniqxe fixed point of $\mathcal{J}$ with the property that there exists $\ell \in (0, \infty)$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - \mathcal{V}(x_1), \cdots, f(x_k) - \mathcal{V}(x_k))\|_k \leq \ell$$

for all $x_1, \cdots, x_k \in \mathcal{A}$. Therefore, $\mathcal{V}$ is viginti a mapping. □

The idea of th following example is taken from [15] which provides an counter-example for the previous theorem.

Example 4.2. *Consider the function*

$$\phi(x) = \begin{cases} x^{20}, & |x| < 1 \\ 1, & |x| \geq 1 \end{cases}$$

where $\phi : \mathbb{R} \longrightarrow \mathbb{R}$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \sum_{n=0}^{\infty} 2^{-20n} \phi(2^n x) \tag{4.38}$$

for all $x \in \mathbb{R}$. Suppose that the function f defined in (4.38) satisfies the functional inequality

$$|\mathcal{G}f(x,y)| \leq \frac{2432902008000000.(1048576)^2}{1048575} \delta \tag{4.39}$$

for all $x, y \in \mathbb{R}$. We show that there do not exist an viginti function $\mathcal{V} : \mathbb{R} \longrightarrow \mathbb{R}$ and a constant $\beta > 0$ such that $|f(x) - \mathcal{V}(x)| \leq \beta |x|^{20}$ for all $x \in \mathbb{R}$. We have

$$|f(x)| = \left| \sum_{n=0}^{\infty} 2^{-20n} \phi(2^n x) \right| \leq \sum_{n=0}^{\infty} \frac{1}{2^{20n}} = \frac{1048576}{1048575}.$$

So, we see that f bounded by $\frac{1048576}{1048575}$ on $\mathbb{R}$. Now, set $0 < \delta < \frac{1}{2^{20}}$. Then, there exists a positive integer k such that $\frac{1}{(2^{20})^{k+1}} \leq \delta < \frac{1}{(2^{20})^k}$

and

$$\begin{aligned}
&2^n(x+10y), 2^n(x+9y), 2^n f(x+8y), 2^n f(x+7y), 2^n f(x+6y),\\
&2^n(x+5y), 2^n(x+4y), 2^n(x+3y), 2^n(x+2y), 2^n(x+y), 2^n(x),\\
&2^n(x-y), 2^n(x-2y), 2^n(x-3y), 2^n(x-4y), 2^n(x-5y),\\
&2^n(x-6y), 2^n f(x-7y), 2^n(x-8y), 2^n(x-9y), 2^n(x-10y),\\
&2^n f(y) \in (-1,1)
\end{aligned}$$

for all $n = 0, 1, 2, \cdots, k-1$. *Hence, for* $n = 0, 1, 2, \cdots, k-1$,

$$\begin{aligned}
&\phi\left(2^n(x+10y)\right) - 20\phi\left(2^n(x+9y)\right) + 190\phi\left(2^n(x+8y)\right)\\
&- 1140\phi\left(2^n(x+7y)\right) + 4845\phi\left(2^n(x+6y)\right) - 15504\phi\left(2^n(x+5y)\right)\\
&+ 38760\phi\left(2^n(x+4y)\right) - 77520\phi\left(2^n(x+3y)\right) + 125970\phi\left(2^n(x+2y)\right)\\
&- 167960\phi\left(2^n(x+y)\right) + 184756\phi\left(2^n(x)\right) - 167960\phi\left(2^n(x-y)\right)\\
&+ 125970\phi\left(2^n(x-2y)\right) - 77520\phi\left(2^n(x-3y)\right) + 38760\left(2^n(x-4y)\right)\\
&- 15504\phi\left(2^n(x-5y)\right) + 4845\phi\left(2^n(x-6y)\right) - 1140\phi\left(2^n(x-7y)\right)\\
&+ 190\phi\left(2^n(x-8y)\right) - 20\phi\left(2^n(x-9y)\right) + \phi\left(2^n(x-10y)\right) - 20!\phi\left(2^n(y)\right).
\end{aligned}$$

From the definition of f *and the inequality, we obtain that*

$$\begin{aligned}
&|\mathcal{G}f(x,y)| =\\
&\Bigg\| \sum_{n=0}^{\infty} 2^{-20n}\phi\left(2^n(x+10y)\right) - 20\sum_{n=0}^{\infty} 2^{-20n}\phi\left(2^n(x+9y)\right)\\
&\quad + 190\sum_{n=0}^{\infty} 2^{-20n}\phi\left(2^n(x+8y)\right) - 1140\sum_{n=0}^{\infty} 2^{-20n}\phi\left(2^n(x+7y)\right)\\
&\quad + 4845\sum_{n=0}^{\infty} 2^{-20n}\phi\left(2^n(x+6y)\right) - 15504\sum_{n=0}^{\infty} 2^{-20n}\phi\left(2^n(x+5y)\right)\\
&\quad + 38760\sum_{n=0}^{\infty} 2^{-20n}\phi\left(2^n(x+4y)\right) - 77520\sum_{n=0}^{\infty} 2^{-20n}\phi\left(2^n(x+3y)\right)\\
&\quad + 125970\sum_{n=0}^{\infty} 2^{-20n}\phi\left(2^n(x+2y)\right) - 167960\sum_{n=0}^{\infty} 2^{-20n}\phi\left(2^n(x+y)\right)\\
&\quad + 184756\sum_{n=0}^{\infty} 2^{-20n}\phi\left(2^n(x)\right) - 167960\sum_{n=0}^{\infty} 2^{-20n}\phi\left(2^n(x-y)\right)
\end{aligned}$$

$$+ 125970 \sum_{n=0}^{\infty} 2^{-20n} \phi\left(2^n (x-2y)\right) - 77520 \sum_{n=0}^{\infty} 2^{-20n} \phi\left(2^n (x-3y)\right)$$
$$+ 38760 \sum_{n=0}^{\infty} 2^{-20n} \left(2^n (x-4y)\right) - 15504 \sum_{n=0}^{\infty} 2^{-20n} \phi\left(2^n (x-5y)\right)$$
$$+ 4845 \sum_{n=0}^{\infty} 2^{-20n} \phi\left(2^n (x-6y)\right) - 1140 \sum_{n=0}^{\infty} 2^{-20n} \phi\left(2^n (x-7y)\right)$$
$$+ 190 \sum_{n=0}^{\infty} 2^{-20n} \phi\left(2^n (x-8y)\right) - 20 \sum_{n=0}^{\infty} 2^{-20n} \phi\left(2^n (x-9y)\right)$$
$$+ \phi\left(2^n (x-10y)\right) - 20! \sum_{n=0}^{\infty} 2^{-20n} \phi\left(2^n (y)\right) \Bigg\|$$
$$\leq \sum_{n=k}^{\infty} 2^{-20n}.2432902008 \times 10^9 \leq \frac{2432902008 \times 10^9 (1048576)^2}{1048575} \delta.$$

So, f satisfies (4.39) for all $x, y \in \mathbb{R}$. we prove that the functional equation (1.1) is not stable. Suppose on the contrary that there exists an viginti function $\mathcal{V} : \mathbb{R} \longrightarrow \mathbb{R}$ and a constant $\beta > 0$ such that $|f(x) - \mathcal{V}(x)| \leq \beta |x|^{20}$, for all $x \in \mathbb{R}$. Then, there exists a constant $c \in \mathbb{R}$ such that $\mathcal{V} = cx^{20}$ for all rational numbers x. So, we arrive that

$$|f(x)| \leq \beta + |c| \,.\, |x|^{20} \tag{4.40}$$

for all $x \in \mathbb{Q}$. Take $m \in \mathbb{N}$ with $m+1 > \beta + |c|$. If x is a rational number in $(0, 2^{-m})$, then $2^n x \in (0,1)$ for all $n = 0, 1, 2, \cdots, m$, and for this x, we get

$$f(x) = \sum_{n=k}^{\infty} 2^{-20} \phi(2^n x)^{20} = (m+1)x^{20} > \beta + |c| \,.x^{20}$$

which contradicts (4.40). Therefore, the functional equation (1.1) is not stable.

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Department of Mathematics
Sacred Heart College
Tirupattur-635601, TamilNadu, India
E-mail: shcrmurali@yahoo.co.in

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Department of Mathematics
Garmsar Branch
Islamic Azad University, Garmsar, Iran
E-mail: abasalt.bodaghi@gmail.com

***

Department of Mathematics
Sacred Heart College
Tirupattur-635601, TamilNadu, India
E-mail: antoyellow92@gmail.com